On the random-neighbor Olami-Feder-Christensen slip-stick model

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We reconsider the treatment of Lise and Jensen (Phys. Rev. Lett. **76**, 2326 (1996)) on the random neighbor Olami-Feder-Christensen stik-slip model, and examine the strong dependence of the results on the approximations used for the distribution of states $p(E)$.

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The work of Olami, Feder and Christensen [1] on a slip-stick earthquake model indicated, some time ago, that self-organized criticality may occur without a local conservation law. Recently, it has been claimed by Lise and Jensen [2] that the random-neighbor version of the OFC model also presents critical behavior above some critical dissipation level $\alpha_c < \alpha_0$, where $\alpha_0 = 1/q$ is associated with local conservation for q neighbors. These authors based their claims on some theoretical mean field arguments and on numerical simulations with systems with up to $N = 400^2$ sites. In order to perform the mean-field calculation they had to make many different assumptions about the behavior of the model.

More recently, Chabanol and Hakin [3], and Bröker and Grassberger [4] performed a more detailed analysis of the same model, showing that what has been interpreted as a critical behavior in [2] indeed corresponds to a subcritical region with very large (but finite) mean avalanche sizes. Although Bröker and Grassberger [4] gives a comprehensive treatment of the random-neighbor version of the OFC model (which we will designate R-OFC), it may be of interest to detect exactly where the theoretical arguments given in [2] fail, since that point is not transparent in their paper and similar problems may occur or be of interest in the future. This is the aim of our paper. We will show that the problem is not in the method used in [2] (which eventually can give useful informations about the mechanism behind SOC) but in the strong dependence of the output of the calculations on the exact form of the distribution of states $p(E)$ of the system.

To reinforce the strong dependence of the results on the specific form of $p(E)$, we revisit the R-OFC model, but this time introducing a simple and small modification on the $p(E)$ distribution, that consists in replacing the interval $[0, E_c]$ where the uniform distribution used by Lise and Jensen was defined by the interval $[0, E^\star]$, with $E^\star < E_c$ (E_c is the threshold value above which the sites become unstable and relax), that is

$$p(E) = \frac{1}{E^\star} \, \Theta(E) \, \Theta(E^\star - E), \tag{1}$$

where $\Theta(x)$ is the Heaviside function (see Figure (1-a)).

The random version of the OFC model (R-OFC) consists of N sites initially with an energy $E_i < E_c$, for $i = 1, ..., N$. The sites with energy E bellow E_c are stable sites (inative) and will be labelled by the superscript $-$; the sites with energy E above E_c are unstable (active) and will be labelled by the superscript $+$. The energies of all sites are increased slowly until the instant t when the energy of a certain site i reaches the value E_c. This site becomes then unstable and the system relaxes in a very short time scale according to the following rules:

$$\begin{aligned} E_i(t + 1) &= 0, \\ E_{rn}(t + 1) &= E_{rn}(t) + \alpha E^+, \end{aligned} \tag{2}$$

where E_{rn} stands for the energy of q other sites chosen at random, and $\alpha \le 1/q$. Eventually some of these q sites may become unstable, relax also, and so may generate an avalanching process that only stops when the energies of all sites are again bellow E_c. If we have $\alpha = \alpha_c = 1/q$ we say that the system is conservative.

Following [2], the probability of an inactive site to be activated by receiving a contribution αE^+ of an active site is

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$$P_+(E^+) = \frac{\int_{E_c-\alpha E^+}^{E_c} p(E)dE}{\int_0^\infty p(E)dE}$$

$$= \frac{E^\star - E_c + \alpha E^+}{E^\star}. \tag{3}$$

The branching ratio σ is the average number of new unstable sites created by a unstable site that relaxes. Clearly, in order to have a critical branching process, we must have $\sigma = 1$. For q random-neighbors, we have

$$\sigma = q \, \frac{\int_{E_c}^\infty dE^+ \, P_+(E^+) \, p(E^+)}{\int_{E_c}^\infty dE^+ \, p(E^+)}, \tag{4}$$

where $p(E^+)$ is the distribution of states for unstable sites, that is, sites with energy above $E^\star$. Adopting the notation $\langle\ldots\rangle \equiv \frac{\int (\ldots) \, p(E) \, dE}{\int p(E) \, dE}$ for averages, we find

$$\sigma = q \left[\frac{E^\star - E_c}{E^\star} + \frac{\alpha}{E^\star} \langle E^+ \rangle \right]. \tag{5}$$

The critical branching ratio $\sigma = 1$ defines a value α_c above which infinite avalanches may occur. The quantity $\langle E^+ \rangle$ is estimated [2] as

$$\langle E^+ \rangle = \frac{\langle E^- \rangle}{1 - \alpha}, \tag{6}$$

The average on E^- is calculated as ,

$$\langle E^- \rangle = \frac{\int_{E_c-\alpha E^+}^{E_c} p(E) \, dE}{\int_{E_c-\alpha E^+}^{E_c} p(E) \, dE}$$

$$= \frac{1}{E^\star - E_c + \alpha \, \langle E^+ \rangle} \int_{E_c-\alpha\langle E^+\rangle}^{E^\star} E^- dE$$

$$= \frac{1}{2} \frac{(E^\star)^2 - E_c^2 + (2E_c - \alpha \, \langle E^+ \rangle) \, \alpha \, \langle E^+ \rangle}{E^\star - E_c + \alpha \, \langle E^+ \rangle}. \tag{7}$$

With this result and Eq. (6), we get

$$(2\alpha - \alpha^2)x^2 + [2E^\star(1-\alpha) - 2E_c] \, x + E_c^2 - (E^\star)^2 = 0, \tag{8}$$

where $x = \langle E^+ \rangle$. This leads to the simple solution

$$\langle E^+ \rangle = \frac{E_c + E^\star}{2 - \alpha}. \tag{9}$$

Finally, we get for the branching ratio

$$\sigma = q \left[\frac{E^\star - E_c}{E^\star} + \frac{\alpha(E_c + E^\star)}{E^\star(2 - \alpha)} \right]. \tag{10}$$

The critical condition $\sigma = 1$ leads to

$$\alpha_c = \frac{E_c - E^\star(q-1)/q}{E_c + E^\star/(2q)}. \tag{11}$$

For example, if $q = 4$ (case studied in [2]), we have

$$\alpha_c = \frac{E_c - 3E^\star/4}{E_c + E^\star/8}. \tag{12}$$

If we consider $E^\star = E_c$ we recover the Lise and Jensen value $\alpha_c = 2/9$. But the value of α_c has a strong dependence on the value of $E^\star$. We see that, already for the value $E^\star/E_c = 24/25 = 0.96$, α_c achieves the physical limit 0.25.

In Figure 2 we show the behavior of the normalized coupling $\tilde\alpha_c = q\alpha_c/4$ as a function of $E^\star$. With this normalization, the conservative case always corresponds to $\tilde\alpha = 0.25$ (as for $q = 4$). We see that the allowed region of values for $E^\star$ so that $\tilde\alpha_c < 0.25$ is very narrow for any q, and that the value of α_c varies strongly in this region. We have already shown in another work (see [5]) that already with lattices with $N = 600^2$ sites it is possible to see a finite mean size avalanche for $\alpha = 0.23$, contrasting with the results of [2] based in simulations in lattices with $N = 400^2$.

Besides showing that Lise and Jensen's approach is not robust with respect to $p(E)$, we may ask about what kind of model produces the uniform distribution

$$p(E) = \Theta(E)\Theta(1 - E) \tag{13}$$

used by those authors. We found that an extremal version of the Feder and Feder model, hereforth called EFF model, indeed produces this distribution. Extremal here is used in the same sense it was first used in the Bak-Snappen co-evolution model [6]. In the dynamics of extremal models there is no driving step. We locate the site i with the largest value of $E_i = \max\{E_j\}$ at the initial instant t. This site relaxes following the original FF rules:

$$E_i^m(t+1) = \eta, \tag{14}$$
$$E_{nn}(t+1) = E_{nn}(t) + \alpha,$$

where η is a noise, $\eta \in [0, \epsilon]$. If we consider $\epsilon = \alpha = 1/4$ we will have (13). In this model the size of an avalanche is defined as the number of sites with energy $E_i^m > 1$ that relaxes in a row. Now, if we repeat the Lise and Jensen calculation using these EFF rules instead of the OFC rules, we get the self-consistent result $\alpha_c = E_c/q$.

It is also possible to show that a more realistic assumption about $p(E)$ leads to essentially the same results obtained by [3,4]. If we simulate the R-OFC model with $q = 4$ we will get a energy distribution $p(E)$ with four peaks [2]. They show clearly that $p(E)$ is not a simple constant. We then decided to repeat the same calculations but supposing this time that $p(E)$ had the (more realistic) form shown in figure (1-b), where Δ_p is half the width of each peak and Δ_b is the width of the gaps between two peaks. That means

$$p(E) = \begin{cases} a, & \text{for } E \in I_1 \text{ or } E \in I_2 \text{ or } E \in I_3 \text{ or } E \in I_4 \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

where $I_1 = [0, \Delta_p]$, $I_2 = [\Delta_p + \Delta_b, 3\Delta_p + \Delta_b]$, $I_3 = [3\Delta_p + 2\Delta_b, 5\Delta_p + 2\Delta_b]$, and $I_4 = [5\Delta_p + 3\Delta_b, 7\Delta_p + 3\Delta_b]$.
We also have that $E^* = 3\Delta_b + 7\Delta_p$ is the maximum value for which $p(E) \neq 0$.
Then we have

$$P_+\left(E^+\right) = \frac{\int_{E_c - \alpha E^+}^{E_c} p(E)\,dE}{\int_0^\infty p(E)\,dE} = \frac{1}{7\,a\Delta_p} \tag{16}$$

The lower limit of the integral in the numerator, $E_c - \alpha E^+$, can now belong to any of the four intervals that define the peaks of the distribution, to which we will asssign the indices $i = 1, 2, 3, 4$. Considering each one of the possibilities, the integrals $P_+^i\left(E^+\right)$ assume the generic form

$$P_+^i\left(E^+\right) = 1 + \frac{(i-1)\Delta_b}{7\Delta_p} - \frac{E_c}{7\Delta_p} + \frac{\alpha E^+}{7\Delta_p}. \tag{17}$$

The branching rate is given by

$$\sigma = 4P_+^i = 4\left[1 + \frac{(i-1)\Delta_b}{7\Delta_p} - \frac{E_c}{7\Delta_p} + \frac{\alpha\langle E^+\rangle}{7\Delta_p}\right]. \tag{18}$$

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In a similar way used to obtain eq. (7), we calculate an expression for $\langle E^- \rangle$, which is associated with eq. (6) to lead

$$\langle E^+ \rangle^i = \frac{E_c}{\alpha (2 - \alpha)} - \frac{[7\Delta_p + (i-1)\Delta_b](1-\alpha)}{\alpha(2-\alpha)} \pm \frac{\sqrt{y_i}}{2\alpha(2-\alpha)}, \tag{19}$$

where

$$y_i = 4\{E_c(1-\alpha) - [7\Delta_p + (i-1)\Delta_b]\}^2 + 4\alpha(2-\alpha)[x_i - 14(i-1)]\Delta_p \Delta_b, \tag{20}$$

with $x_i = 24, 26, 32$ and 42, for $i = 1, 2, 3$ and 4, respectively. Imposing the branching condition $\sigma = 1$ and using eq. (19) we get

$$7\Delta_p(2+\alpha) + 4(i-1)\Delta_b - 4E_c(1-\alpha) + 7\alpha E_c \pm \sqrt{y_i} = 0 \tag{21}$$

For instance, if we take $\Delta_p = 0.08$ and $\Delta_b = 0.1$ (that corresponds to Figure (1-b)), the critical branching condition leads to values of α^* outside the physical range (that is, $\alpha^* > 1/4$). Therefore, in this particular example, it is physically forbidden to assume that $\sigma = 1$, so there is no self-organized critical state.

If we take the limit for the conservative case (that is, $\Delta_p \to 0$ and $\Delta_b \to \alpha E_c$), the four peaks tend to four delta functions at $(i-1)\alpha E_c$, and it is easy to see that the condition $\sigma = 1$ leads to the only possibility $\alpha^* = \alpha_c = 1/4$ (we obtain $\alpha^* > 1/4$ for $i = 1, 2, 3$). It can also be shown that, if we consider the limit $\Delta_b \to 0$ and $\Delta_p \to E_c / 7$, (that is $p(E)$ is constant in the interval $[0, E_c]$ which corresponds to the approximation of ref. [2]) then $\alpha^* = 2/9$.

In general, for all values of i, the regions of the parameter space associated with $\alpha \leq 1/4$ are determined by

$$E_c - \frac{175}{24}\Delta_p - \frac{2x_i}{21}\Delta_b \leq 0. \tag{22}$$

From this inequality plus the relation $E_c \geq 7\Delta_p + 3\Delta_b$, (see Figure 3) we see that only for a very small range of the parameters Δ_p and Δ_b there are values of α_c in the physical range ($0 < \alpha_c \leq 1/4$). In all of those cases, Δ_b is very small and the shape of $p(E)$ is very close to the constant form used by Lise and Jensen. Moreover, α varies strongly in these allowed ranges.

In conclusion, we showed that, besides not having considered lattices big enough, the problem with the approach used by Lise and Jensen in [2] is not in the method itself, but in the strong dependence of the output of the calculations on the compatibility between the distribution $p(E)$ and the assumed dynamical rules which presumably lead to it. We also showed that the $p(E)$ approximation used by Lise and Jensen does not correspond to the model they intend to analyse, namely R-OFC, but to another non-conservative model, designated by us EFF (Extremal Feder and Feder model). If we adopt the EFF's dynamical rules, the Lise and Jensen's method will lead to the right conclusions. In the end, we followed the same approach but now considering a more realistic approach for $p(E)$ and get essentially the same results that had already been obtained through the use of other arguments in [3,4], that is in the R-OFC model there is SOC only in the conservative limit.

[1] Z. Olami, H. J. F. Feder and K. Christensen, Phys. Rev. Lett. **68** 1244 (1992).

[2] S. Lise and H. J. Jensen, Phys. Rev. Lett. **76** 2326 (1996).

[3] M-L. Chabanol and V. Hakin, Phys. Rev. E **56** R2343 (1997).

[4] H-M. Bröker and P. Grassberger, Phys. Rev. E **56** 3944 (1997).

[5] S. T. R. Pinho, C. P. C. Prado and O. Kinouchi, to be published in Physica A, proccedings of the V Latin America Workshop on Non-linear Phenomena .

[6] P. Bak and K. Sneppen, Phys. Rev. Lett **71** 4083 (1993)

Figure Captions

Fig. 1- a) The uniform approximation for the energy distribution $p(E)$ of the R-OFC model. The solid line corresponds to the interval of E used by Lise and Jensen, $[0, E_c]$; the dashed line corresponds to a smaller interval of E, $[0, E^\star]$, used in our calculations. **b)** A more realistic approximation (non-uniform) of the energy distribution of the R-OFC model, for $q = 4$ (with four peaks). The parameters Δ_p and Δ_b are, respectively, the half-width of the peaks and the width of the gaps. The value of $p(E)$ at the peaks is **a** and $E_c \geq E^\star = 7\Delta_p + 3\Delta_b$.

Fig. 2- Normalized 'critical' dissipation level $\tilde{\alpha}_c = k\alpha/4$ as a function of $E^\star$. Values for $\tilde{\alpha}_c$ above 0.25 are not physically admissible.

Fig. 3- Space of parameters for $p(E)$ in terms of $\gamma_p = \Delta_p/E_c$ and $\gamma_b = \Delta_b/E_c$. The shaded regions correspond to the intersection between $\alpha \leq 1/4$ and $(7\Delta_p + 3\Delta_b)/E_c = 7\gamma_p + 3\gamma_b \leq 1$. Depending on the value of $E_c - \alpha E^+$, we have (a) $E_c - \alpha E^+ \, \epsilon \, [0, \Delta_p]$; (b) $E_c - \alpha E^+ \, \epsilon \, [\Delta_p + \Delta_b, 3\Delta_p + \Delta_b]$; (c) $E_c - \alpha E^+ \, \epsilon \, [3\Delta_p + 2\Delta_b, 5\Delta_p + 2\Delta_b]$; and (d) $E_c - \alpha E^+ \, \epsilon \, [5\Delta_p + 3\Delta_b, 7\Delta_p + 3\Delta_b]$. -
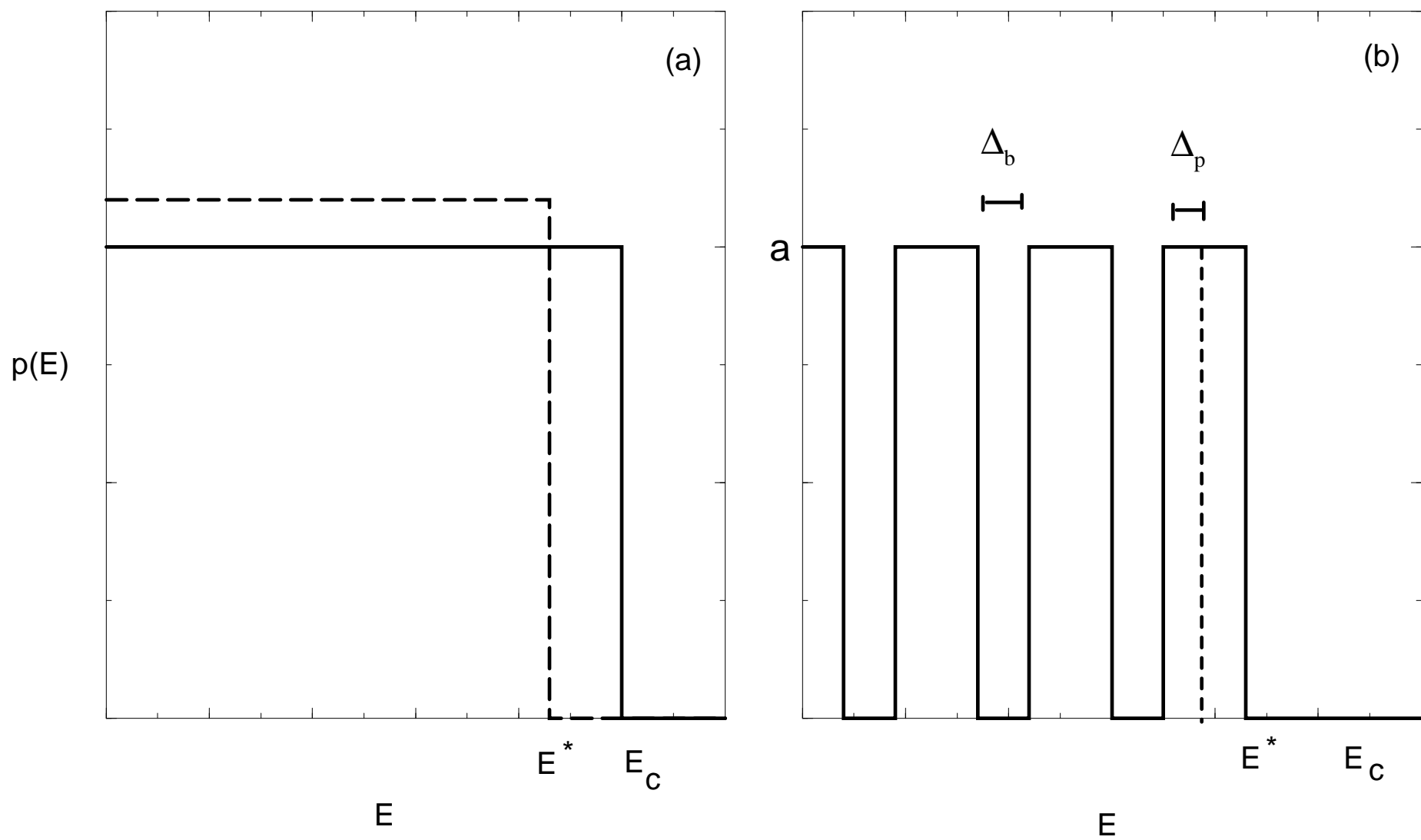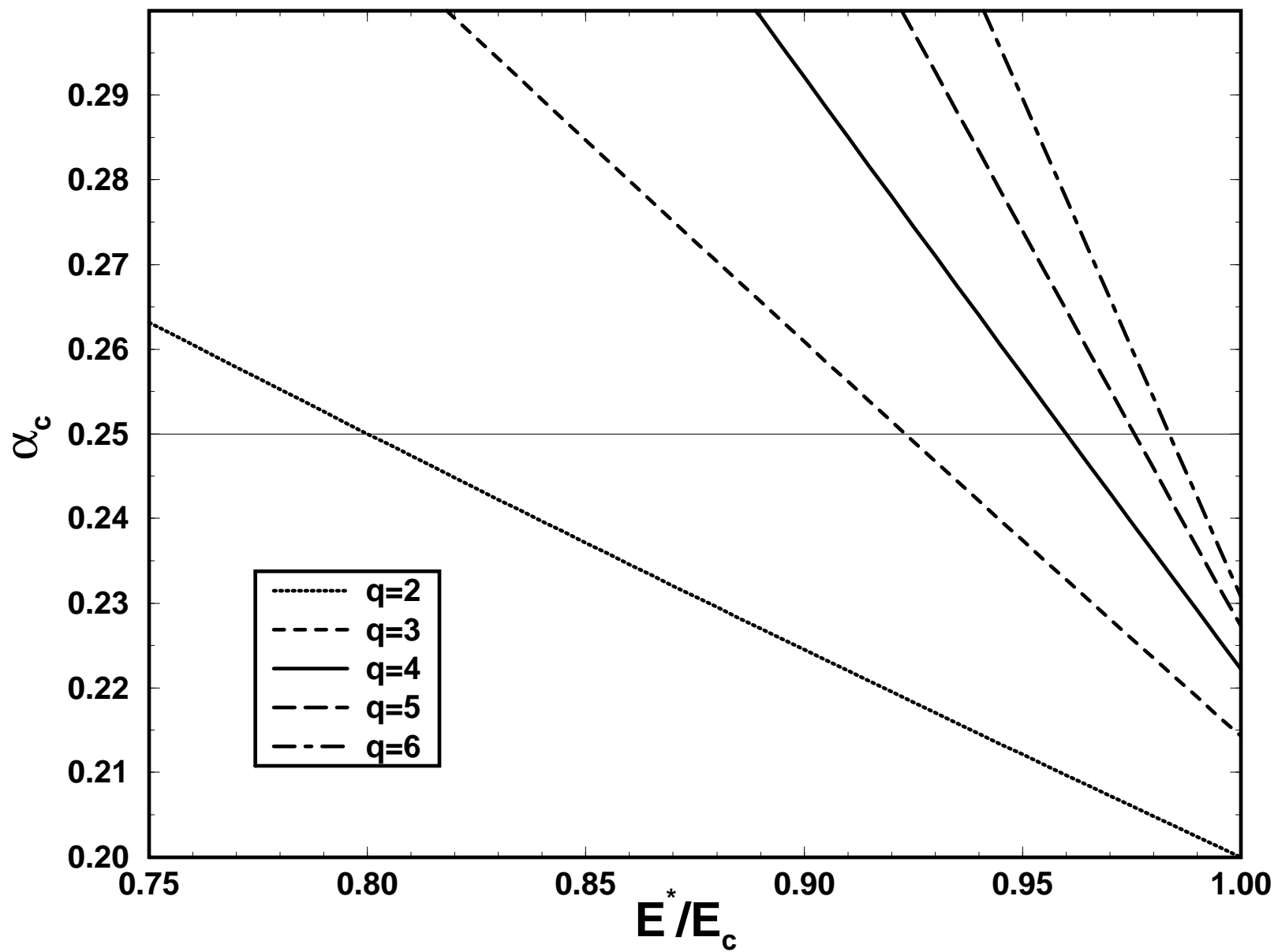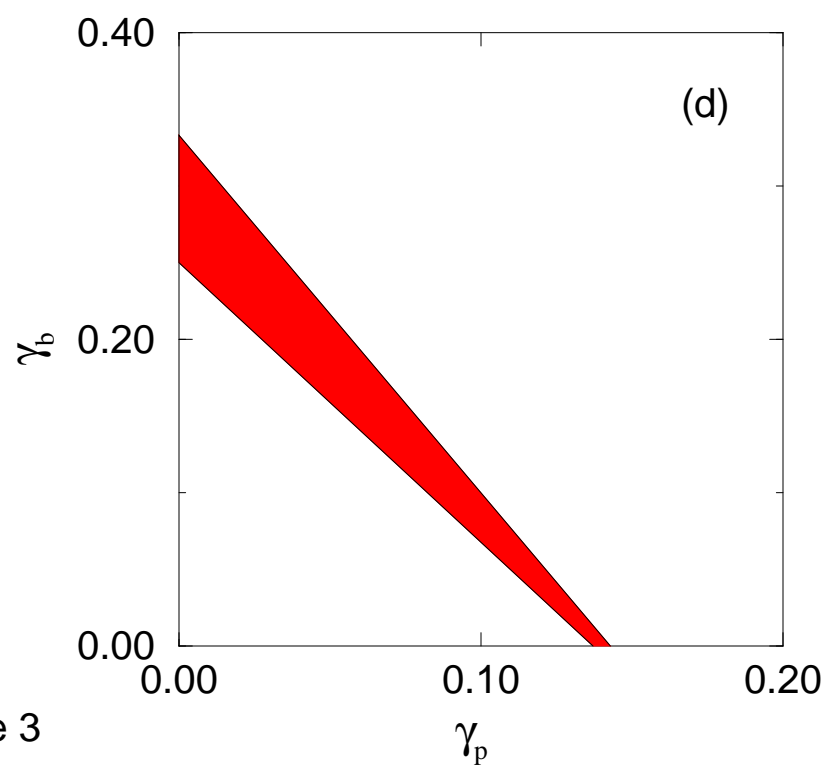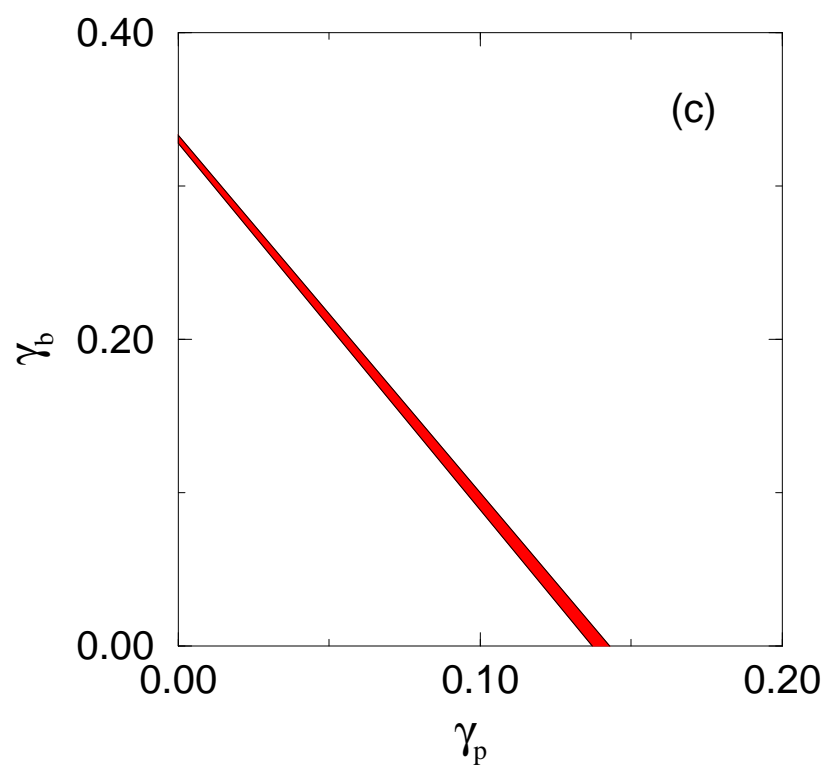
Figure 1

Figure 2

Figure 3